The phase diagram of an exactly solvable *t-J* ladder model

LETTER TO THE EDITOR

The phase diagram of an exactly solvable t–J ladder model

Holger Frahm† and Anjan Kundu‡

† Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany
‡ Saha Institute of Nuclear Physics, Theory Group, 1/AF Bidhan Nagar, Calcutta 700 064, India

E-mail: `anjan@tnp.saha.ernet.in`

Abstract. We study a system of one-dimensional t–J models coupled to a ladder system. A special choice of the interaction between neighbouring rungs leads to an integrable model with supersymmetry, which is broken by the presence of rung interactions. We analyse the spectrum of low-lying excitations and the ground-state phase diagram at zero temperature.

Systems of one-dimensional magnetic or electronic chains coupled to so-called ladders have attracted increasing attention as a scenario in which two-dimensional physics can be studied by starting from well established one-dimensional theories [1–3]. The properties of the undoped systems, i.e. spin ladders, have been thoroughly investigated by a variety of methods [1–6] and recently integrable models with tunable interaction parameters have been introduced [7–13]. To understand the properties of the doped ladder systems the dynamics of the holes created under doping in the background of antiferromagnetically interacting spins has to be studied [14]. This question has been studied within a model of coupled t–J chains forming a ladder-like structure in a series of papers [14–16] with interesting results. Within this approach, however, multi-hole-scattering processes or the dynamics of anti-bonding hole states could not be accessed.

Our aim here is to propose a modified t–J ladder model, which becomes integrable at certain limits and therefore allows us to investigate the properties of this system exactly and in more detail. In the thermodynamic limit, we can analyse further the nature of the ground state and predict the onset of excitations, thereby identifying the phase diagram of the system. The system that we consider consists of two coupled t–J chains labelled $a = 1, 2$. The electrons with spin σ on site j of the chains are described in terms of canonical Fermi operators $c_{j\sigma}^{(a)}$. In terms of these the chain Hamiltonian is

$$H_{t-J}^{(a)} = \sum_j \left\{ H_{t-J}^{(a)} \right\}_{jj+1}$$

with

$$\left\{ H_{t-J}^{(a)} \right\}_{jk} = -t\mathcal{P}\left(\sum_{\sigma=\uparrow\downarrow} c_{\sigma j}^{(a)\dagger} c_{\sigma k}^{(a)} + \text{h.c.} \right)\mathcal{P} + J\left(S_j^{(a)} \cdot S_k^{(a)} - \frac{1}{4} n_j^{(a)} n_k^{(a)} \right) + n_j^{(a)} + n_k^{(a)} \quad (1)$$

where $\mathcal{P}$ projects out double occupancies, the $S_j^{(a)}$ are spin operators on site j, and $n_j^{(a)} = n_{j\uparrow}^{(a)} + n_{j\downarrow}^{(a)}$ is the total number of electrons on site j.

The Hamiltonian of the ladder model that we consider then reads

$$H = \sum_a H_{t-J}^{(a)} + H_{\text{int}} + H_{\text{rung}} - \mu \hat{N}. \quad (2)$$

Here, μ is the chemical potential coupling to the total number of electrons in the system, H_{int} is the interaction between neighbouring rungs of the ladder, given by the expression

$$H_{\text{int}} = -\sum_j \left\{ H_{t-J}^{(1)} \right\}_{jj+1} \left\{ H_{t-J}^{(2)} \right\}_{jj+1}$$

and H_{rung} contains all the interactions present on a single rung of the ladder and reads

$$H_{\text{rung}} = \sum_j \left\{ -t'\mathcal{P}\left(\sum_{\sigma=\uparrow\downarrow} c_{\sigma j}^{(1)\dagger} c_{\sigma j}^{(2)} + \text{h.c.} \right)\mathcal{P} + J'\left(\boldsymbol{S}_j^{(1)} \cdot \boldsymbol{S}_j^{(2)} - \frac{1}{4} n_j^{(1)} n_j^{(2)} \right) + V n_j^{(1)} n_j^{(2)} \right\}.$$

(3)

This is similar to the terms in (1) describing the interaction of electrons on the bonds within a rung but contains independent coupling constants t', J' and an additional Coulomb interaction of strength V. Below we shall use a basis of states on a single rung which are eigenstates to (3): at half-filling (one electron per site) the electrons can form a spin singlet

$$|s\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

or triplet state $|t_+\rangle \equiv |\uparrow\uparrow\rangle$ etc ($|\sigma\tau\rangle = c_\sigma^{(1)\dagger} c_\tau^{(2)\dagger}|00\rangle$). Doping of the system creates the states

$$|\sigma_\pm\rangle \equiv \frac{1}{\sqrt{2}}(|\sigma 0\rangle \pm |0\sigma\rangle)$$

with $\sigma = \uparrow$ or $\downarrow$, describing a single electron with spin σ in a bonding (anti-bonding) state or the Fock vacuum (two-hole state) $|d\rangle \equiv |00\rangle$ on a rung. Below we shall be particularly interested in the strong-coupling regime $J' \gg 1$, $V \gg \mu + |t'|$ near half-filling. This implies that the triplet states $|t_\alpha\rangle$ in the two-particle sector will be energetically unfavourable.

To study the phase diagram of the system at low temperatures we note that by excluding the triplet states and choosing the coupling constants on the legs of the ladder as $J = 2t = 2$ the Hamiltonian (2) can be rewritten as

$$H = -\sum_j \Pi_{jj+1} - \sum_{l=1}^{5} A_l N_l + \text{constant}$$

(4)

where N_l, $l = 1, 2$ (3, 4), is the number of bonding (anti-bonding) single-particle rung states with spin $\uparrow$, $\downarrow$ and N_5 is the number of empty rungs (on the remaining $N_0 = L - \sum_l N_l$ rungs the two electrons form a singlet). The (graded) permutation operator Π_{jk} interchanges the states on rungs j and k giving a minus sign if both rungs are fermionic, i.e. singly occupied. In the restricted Hilbert space that we are considering here we can absorb J' into the definition of V which allows us to express the potentials A_l in terms of the coupling constants in (3) as

$$
\begin{aligned}
A_1 &= A_2 \equiv \mu_+ = t' - \mu + V \\
A_3 &= A_4 \equiv \mu_- = -t' - \mu + V \\
A_5 &\equiv \tilde{V} = -2\mu + V.
\end{aligned}
$$

(5)

The relative strength of these potentials will determine the ground state and the spectrum of low-lying excitations in the ladder.

Note that the Hamiltonian (4) is that of a gl(2|4)-invariant superspin chain in the presence of external symmetry-breaking fields, and the grading of the states $|s\rangle$ and $|d\rangle$ is bosonic while the two doublets $|\sigma_\pm\rangle$ carry fermionic grading. This model is integrable by means of the Bethe *ansatz* (BA) (see e.g. [17–20]). In the strong-coupling regime near half-filling, where the ground state shows little deviation from the dimerized state $\otimes_j |s\rangle_j$, doping will lead to the condensation of either the fermionic single-hole states $|\sigma_\pm\rangle$ or the bosonic double-hole state $|d\rangle$

into the ground state—depending on the relative strength of the chemical potentials μ_-, μ_+ and the rung interaction $\tilde{V}$. For simplicity we shall consider the case where $\mu_+ = \mu_- \equiv \tilde{\mu}$ below, i.e. two degenerate single-particle bands. Due to the grading, different sets of BA equations have to be considered to describe these processes (see e.g. [17,21,22]): since the Bethe *ansatz* states are highest in weight in the gl(2|4) algebra we have to deal separately with two regions in the low-temperature phase diagram of the t–J ladder (2) between half- and quarter-filling, corresponding to $4N_5 < \sum_{l=1}^{4} N_l$ and $4N_5 > \sum_{l=1}^{4} N_l$, respectively.

For the first case we have to order the rung basis as $|s\rangle$, $|\sigma_\pm\rangle$, $|d\rangle$, and the eigenstates of (4) are parametrized by solutions of the BA equations corresponding to this grading (BFFFFB):

$$\left[e_1(\lambda_j^{(1)}) \right]^L = \prod_{k=1}^{M_2} e_1(\lambda_j^{(1)} - \lambda_k^{(2)})$$

$$\prod_{k \neq j}^{M_r} e_2(\lambda_j^{(r)} - \lambda_k^{(r)}) = \prod_{l=1}^{M_{r-1}} e_1(\lambda_j^{(r)} - \lambda_l^{(r-1)}) \prod_{m=1}^{M_{r+1}} e_1(\lambda_j^{(r)} - \lambda_m^{(r+1)}) \qquad r = 2, \ldots, 4 \qquad (6)$$

$$\prod_{k=1}^{M_4} e_1(\lambda_j^{(5)} - \lambda_k^{(4)}) = 1$$

with $e_n(x) \equiv (x + \mathrm{i}n/2)/(x - \mathrm{i}n/2)$. Here the number of roots on the rth level is related to the numbers N_l in (4) by $M_{r+1} = M_r - N_r$ (we define $M_0 \equiv L$). The energy of the corresponding state is

$$E(\{\lambda_j^{(r)}\}) = -2M_1 + \sum_{j=1}^{M_1} \frac{1}{(\lambda_j^{(1)})^2 + 1/4} - \sum_{l=1}^{5} A_l N_l. \qquad (7)$$

In the thermodynamic limit $L \to \infty$ with M_r/L fixed, the solutions of (6) are built on two types of so-called string, namely complexes

$$\lambda_k^{(r-n)} = \xi^{(r)} + \frac{\mathrm{i}}{2}(n - 2k) \qquad k = 0, \ldots, n, \, 0 \leqslant n < r, \text{ and } 2 \leqslant r \leqslant 4$$

corresponding to bound states of holes with different spin and parity with densities $\rho^{(r)}(\xi)$ and

$$\lambda_k^{(r)} = \Lambda_m^{(r)} + \frac{\mathrm{i}}{2}(m + 1 - 2k) \qquad k = 1, \ldots, m, \, 2 \leqslant r \leqslant 4$$

corresponding to bound spin states with densities $\sigma_m^{(r)}(\Lambda)$, $r = 2, 3, 4$. On level 1 and 5 all solutions are real with densities $\sigma_1^{(r)}(\lambda^{(r)})$, $r = 1, 5$.

By minimizing the free energy of the system (see e.g. [18, 20]) one finds that at most the bound states of single-particle rung states $|\sigma_\pm\rangle$ with density $\rho^{(4)}$ and empty rungs $|d\rangle$ with density $\sigma_1^{(5)}$ are present in the ground state. The densities are determined through the Bethe *ansatz* integral equations

$$\rho^{(4)}(x) = a_4(x) - \int_s \mathrm{d}y \, K(x - y)\rho^{(4)}(y) - \int_d \mathrm{d}y \, a_1(x - y)\sigma_1^{(5)}(y)$$

$$\sigma_1^{(5)}(x) = \int_s \mathrm{d}y \, a_1(x - y)\rho^{(4)}(y) \qquad (8)$$

where $K(x) = \sum_{n=1}^{3} a_{2n}(x)$ and $a_n(x) = 2n/(\pi(4x^2 + n^2))$. The corresponding dispersions ϵ_s and ϵ_d are given by similar integral equations:

$$\epsilon_s(x) = 2\pi a_4(x) - 4(2 + \tilde{\mu}) - \int_s \mathrm{d}y \, K(x - y)\epsilon_s(y) + \int_d \mathrm{d}y \, a_1(x - y)\epsilon_d(y)$$

$$\epsilon_d(x) = \tilde{\mu} - \tilde{V} - \int_s \mathrm{d}y \, a_1(x - y)\epsilon_s(y). \qquad (9)$$

The chemical potential and the rung interaction determine the ranges of integration:

$$\int_\alpha \mathrm{d}y = \left\{ \int_{-\infty}^{-Q_\alpha} + \int_{Q_\alpha}^\infty \right\} \mathrm{d}y$$

through the condition $\epsilon_\alpha(\pm Q_\alpha) = 0$. In terms of the solution of (8) the density of holes is

$$n_h = 2 \int_s \mathrm{d}y \, \rho^{(4)}(y) - \int_d \mathrm{d}y \, \sigma_1^{(5)}(y). \tag{10}$$

For $\tilde{V} < \tilde{\mu} < -2$ both ϵ_s and ϵ_d are positive and hence the dimer state is the ground state of the system. For larger chemical potential, single-hole states $|\sigma_\pm\rangle$ with dispersion ϵ_s begin to condense into this ground state. The electronic properties in this phase are those of the degenerate supersymmetric t–J model [18]. For

$$\tilde{V} > \tilde{\mu} - \int_s \mathrm{d}y \, a_1(y)\epsilon_s(y)$$

the system enters a phase where in addition double-hole rung states $|d\rangle$ with energy ϵ_d will be present in the ground state.

For $\tilde{\mu} < \tilde{V}$ we have to order the rung states as $|s\rangle$, $|d\rangle$, $|\sigma_\pm\rangle$ leading to a different set of Bethe *ansatz* equations corresponding to BBFFFF grading ($M_6 \equiv 0$):

$$\left[e_1(\lambda_j^{(1)}) \right]^L = \prod_{k \neq j}^{M_1} e_2(\lambda_j^{(1)} - \lambda_k^{(1)}) \prod_{k=1}^{M_2} e_1(\lambda_k^{(2)} - \lambda_j^{(1)})$$

$$\prod_{l=1}^{M_1} e_1(\lambda_j^{(2)} - \lambda_l^{(1)}) = \prod_{m=1}^{M_3} e_1(\lambda_j^{(2)} - \lambda_m^{(3)}) \tag{11}$$

$$\prod_{k \neq j}^{M_r} e_2(\lambda_j^{(r)} - \lambda_k^{(r)}) = \prod_{l=1}^{M_{r-1}} e_1(\lambda_j^{(r)} - \lambda_l^{(r-1)}) \prod_{m=1}^{M_{r+1}} e_1(\lambda_j^{(r)} - \lambda_m^{(r+1)}) \qquad r = 3, \ldots, 5.$$

(Note that due to the reordering of the basis the number of double-hole rung states is now N_1.) The energy of the corresponding Bethe *ansatz* eigenstates is again given by (7) with $A_1 = \tilde{V}$ and $A_l = \tilde{\mu}$ for $2 \leqslant l \leqslant 5$.

Like for (6) the solutions of these equations can be classified in strings with densities $\sigma_m^{(r)}$ for positive integers m and $r = 1, 3, 4, 5$ and single-hole bound states with densities $\rho^{(r)}$, $r = 3, 4, 5$. Only real roots with density $\sigma_1^{(2)}$ are possible on the second level of (11). The ground-state configuration is completely determined by

$$\rho^{(5)}(x) = -\int_s \mathrm{d}y \, K(x-y)\rho^{(5)}(y) + \int_d \mathrm{d}y \, a_4(x-y)\sigma_1^{(1)}(y)$$

$$\sigma_1^{(1)}(x) = a_1(x) + \int_s \mathrm{d}y \, a_4(x-y)\rho^{(5)}(y) - \int_d \mathrm{d}y \, a_2(x-y)\sigma_1^{(1)}(y). \tag{12}$$

The dispersions of these excitations are determined through the integral equations

$$\epsilon_s(x) = 4(\tilde{V} - \tilde{\mu}) - \int_s \mathrm{d}y \, K(x-y)\epsilon_s(y) + \int_d \mathrm{d}y \, a_4(x-y)\epsilon_d(y)$$

$$\epsilon_d(x) = 2\pi a_1(x) - (2 + \tilde{V}) + \int_s \mathrm{d}y \, a_4(x-y)\epsilon_s(y) - \int_d \mathrm{d}y \, a_2(x-y)\epsilon_d(y) \tag{13}$$

and the density of holes is

$$n_h = \int_d \mathrm{d}y \, \sigma_1^{(1)}(y) - 2 \int_s \mathrm{d}y \, \rho^{(5)}(y).$$

With these equations the phase diagram of the t–J ladder (4) can be completed: for $\tilde{\mu} < \tilde{V} < -2$ the system is in the dimer phase; on increasing the rung interaction $\tilde{V}$ the system

enters a phase where the ground state is a Fermi sea of double-hole states with dispersion ϵ_d moving in the background of dimer states $|s\rangle$. On increasing $\tilde{\mu}$ in this phase beyond $\frac{1}{8}(7\tilde{V}-2)$, both double- and single-hole rung states appear in the ground-state configuration.

Numerical integration of the integral equations for the densities and dressed energies allows us to determine the phase boundaries as a function of the hole density n_h and the rung interaction $V = 2\tilde{\mu} - \tilde{V}$ in equation (3). In figure 1 the resulting phase diagram is shown. For $V \gtrsim 2$, condensation of the single-hole states $|\sigma_\pm\rangle$ is energetically favourable. For sufficiently strong *attractive* rung interaction, however, doping of the system first introduces double-hole rung states $|d\rangle$. Interestingly, we find that for $2 > V > 2 - \frac{1}{2}(\psi(\frac{7}{8}) - \psi(\frac{7}{8})) \approx 1.676$ ($\psi(x)$ is the digamma function), increasing the hole concentration can drive the system from the latter phase into the one without double-hole rung states.
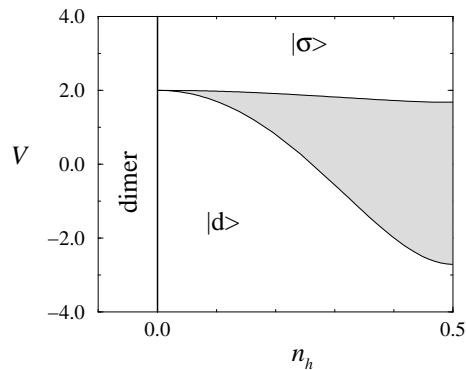
Figure 1. The phase diagram of the t–J ladder with rung interaction (3) for $t' = 0$ and $J' \gg 1$ as a function of the hole concentration n_h: for large repulsive V the ground state is a Fermi sea of single-hole states $|\sigma_\pm\rangle$ propagating in the background of rung dimers. For sufficiently strong attractive rung interaction, the double-hole rung states $|d\rangle$ are energetically favourable. In the intermediate region (shaded), dimers *and* both types of hole rung state coexist in the ground-state configuration.

In summary we have studied the phase diagram of a system of coupled t–J models in the framework of an integrable system. By choosing properly the interaction between the neighbouring rungs, the resulting ladder model can be brought into the form of a gl(2|4)-invariant exchange model in the presence of symmetry-breaking fields, which enter the Hamiltonian as kinetic and interaction terms on a single rung. The spectrum of excitations and the zero-temperature phase diagram have been determined by means of the Bethe *ansatz*. Extensions are possible by assigning different energies to the single-particle rung states of different parity ($t' \neq 0$) and including the triplet rung states (J' finite). As has been shown for the undoped case, new phases can be expected for finite J' [10].

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